EE-565 – W1
ELECTROMECHANICAL ENERGY CONVERSION

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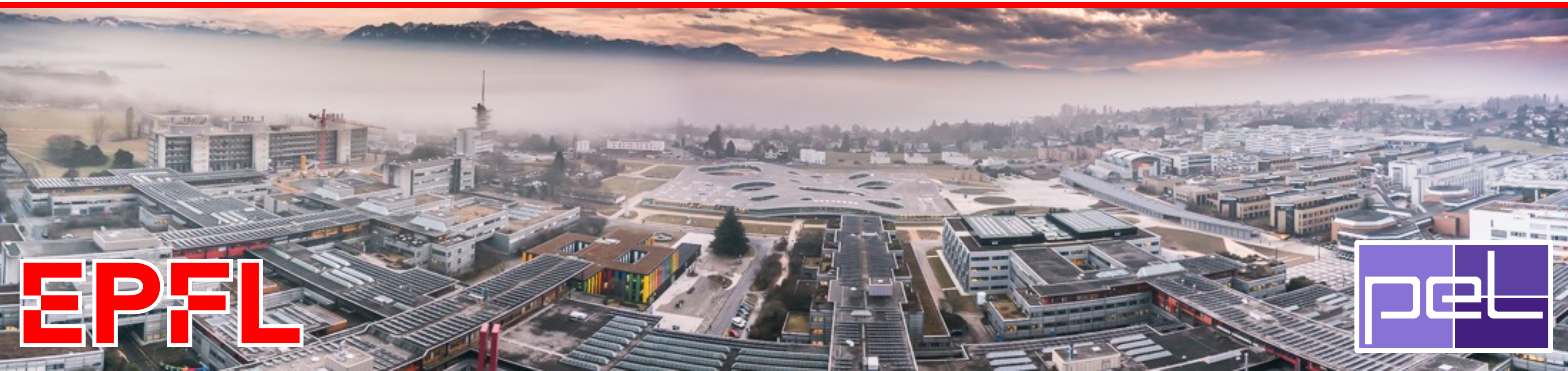


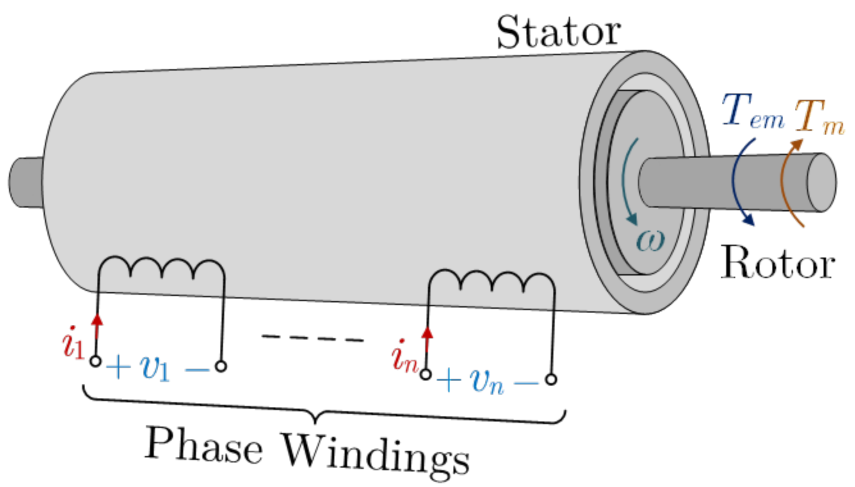
FUNDAMENTALS OF ELECTROMECHANICAL CONVERSION THEORY

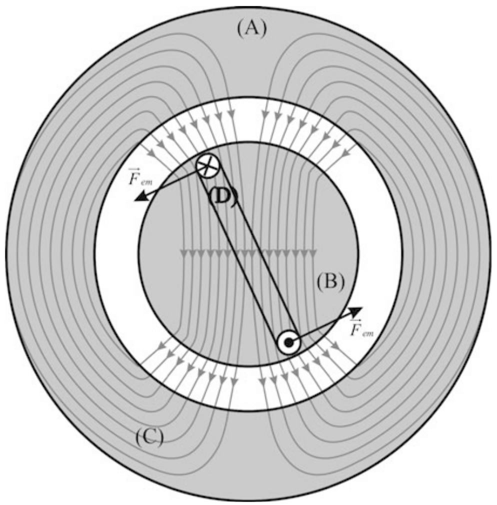
Electrical Equations, Mechanical Equations, Electromagnetic Energy and Co-energy

MODEL OF A ROTATING MACHINE

We will mainly consider electrical machines of cylindrical shape:

- The stator form usually resembles that of a hollow cylinder
- The rotor form can vary for different types, and is inserted into the stator, and can rotate
- The rotor shaft extends out of the machine and serves as a mechanical connection of the work machine
- Both the stator and rotor are made of ferromagnetic material and separated by an air gap
- Both the stator and rotor contain windings and/or permanent magnets that create fields
- Electromagnetic torque is created due to the interaction between the stator and rotor field
- This electromagnetic torque acts on the rotor and turns it into rotation





ELECTRICAL EQUATIONS

Faraday-Neumann Law

$$\oint_\gamma \vec{E}\cdot\hat{t}\cdot dl = -\iint_{S_\gamma} \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\cdot dS$$

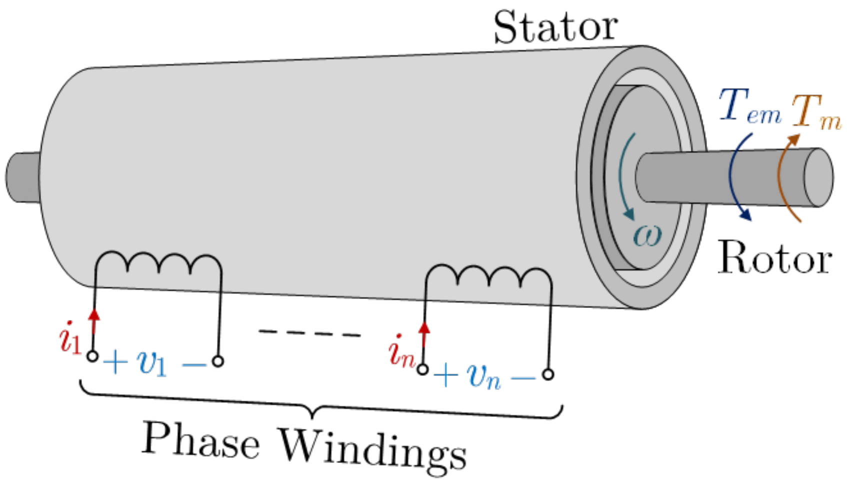
$$\underbrace{v_k}_{\text{Winding Voltage}} = \underbrace{R_k \cdot \overbrace{i_k}^{\text{Winding Current}}}_{\text{Resistive Drop}} + \underbrace{\frac{\overbrace{\mathrm{d}\phi_k}^{\text{Flux Linkage}}}{\mathrm{d}t}}_{\text{Back-EMF}}$$

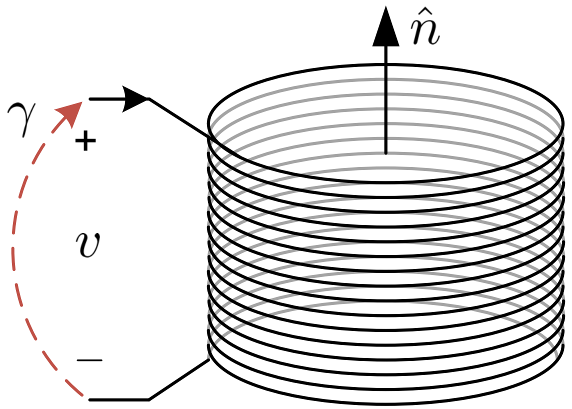
(for each *k*-th winding)

(Note: the “–” sign is canceled by using passive sign notation)

$$\boldsymbol{v} = \boldsymbol{R}\cdot\boldsymbol{i} + \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t}$$

(in vector form)



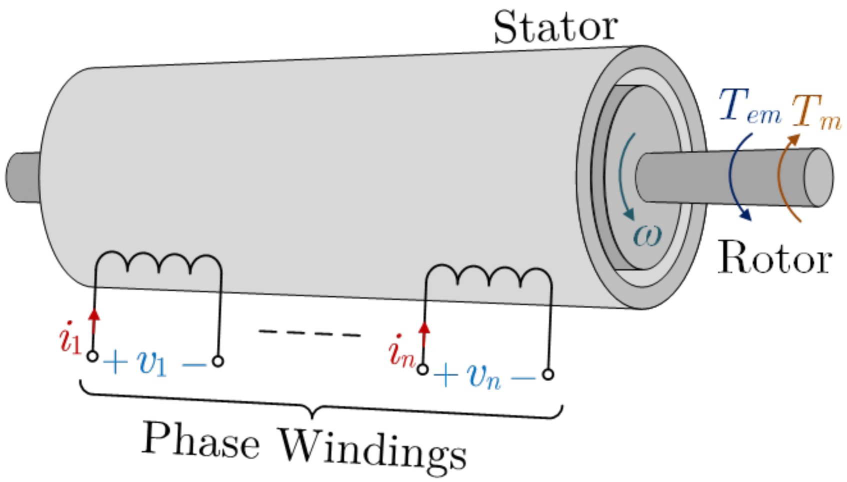


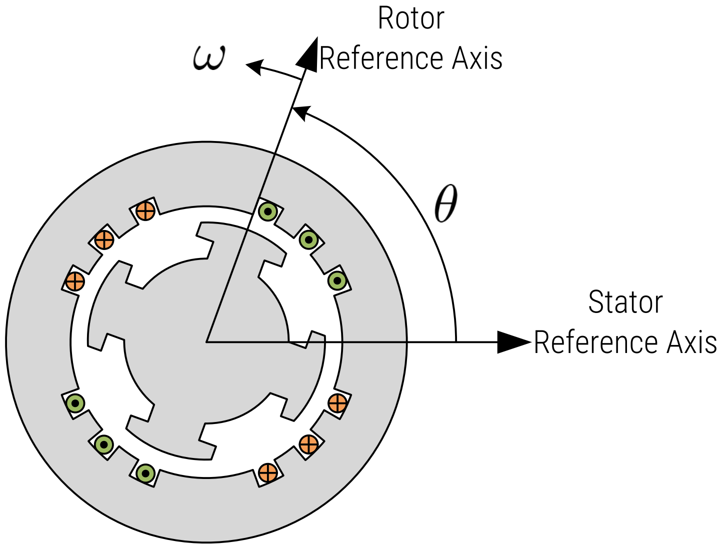
MECHANICAL EQUATIONS

$$\underbrace{\frac{\mathrm{d}\overbrace{\theta}^{\text{Mechanical Angle}}}{\mathrm{d}t} = \underbrace{\omega}_{\text{Angular Speed}}}$$

$$\underbrace{\overbrace{J}^{\text{Moment of Inertia}} \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t}}_{\text{Angular Acceleration}} + \underbrace{\overbrace{F(\omega)}^{\text{Friction Coefficient}} \cdot \omega}_{\text{Friction}} = \overbrace{T_{em}}^{\text{Electromagnetic Torque}} - \underbrace{T_m}_{\text{Loading/Braking Torque}}$$

(Note: positive electromagnetic torque if “motoring” mode)





ELECTRIC MACHINE AS ELECTROMECHANICAL CONVERTER

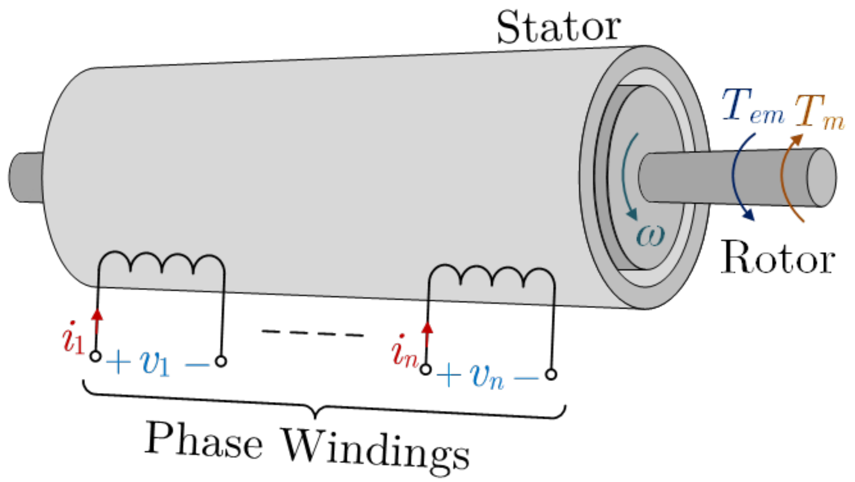
Dynamical model of the machine

$$\boldsymbol{v} = \boldsymbol{R} \cdot \boldsymbol{i} + \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$$

$$J \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t} + F(\omega) \cdot \omega = T_{em} - T_m$$

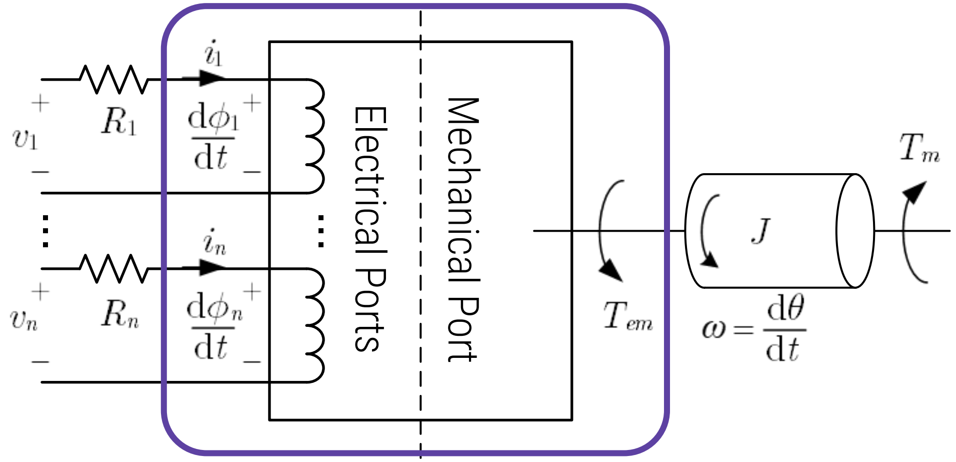
? Unknown



Fluxes, **Rotor Angle** and **Rotor Angular Speed** are the **State Variables**

Voltages and **Loading Torque** are the **System Inputs**

Currents and **Electromagnetic Torque** are the **System Outputs**



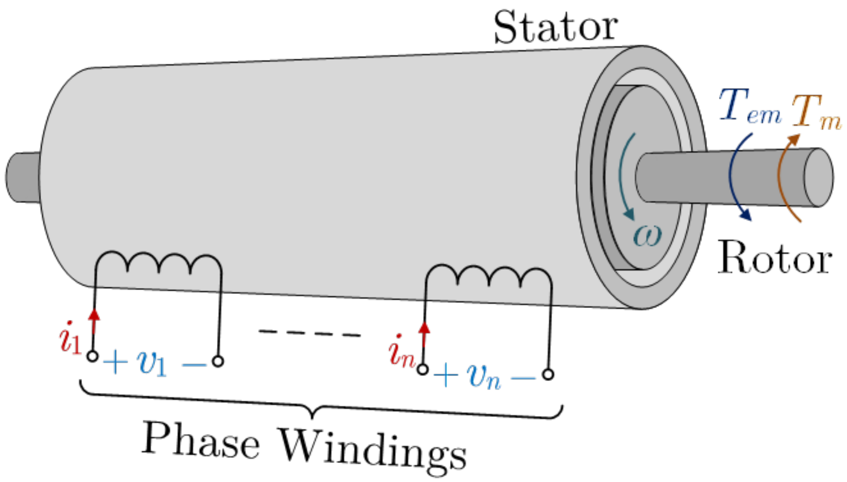
We need to find expressions for the fluxes and for the electromagnetic torque

$$\phi_1 = \phi_1(?) \quad \cdots \quad \phi_n = \phi_n(?) \qquad T_{em} = T_{em}(?)$$

ENERGY-BASED ELECTROMECHANICAL MODEL

By energy conservation principle

$$\underbrace{P_{in}}_{\text{Power Input}} = \underbrace{P_{out}}_{\text{Power Output}} + \underbrace{\frac{\mathrm{d}W}{\mathrm{d}t}}_{\text{Change of Stored Energy}} + \underbrace{P_{loss}}_{\text{Internal Losses}}$$



Power Input (from electrical ports)

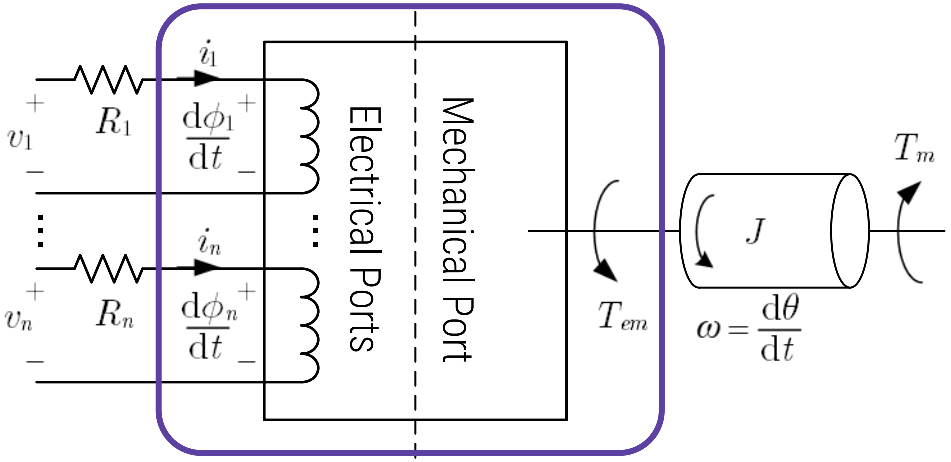
$$P_{in} = i_1 \cdot \frac{\mathrm{d}\phi_1}{\mathrm{d}t} + \cdots + i_n \cdot \frac{\mathrm{d}\phi_n}{\mathrm{d}t}$$

$$= \sum_{k=1}^{N} i_k \cdot \frac{\mathrm{d}\phi_k}{\mathrm{d}t} = \boldsymbol{i}^{\mathrm{T}} \cdot \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t}$$

Internal stored energy is only Electromagnetic Energy $W = W_{em}$

Power Output (from mechanical port)

$$P_{out} = T_{em} \cdot \omega = T_{em} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$



Internal losses are neglected (they can be added in a second modeling stage)

ENERGY-BASED ELECTROMECHANICAL MODEL

By energy conservation principle

$$\frac{\mathrm{d}W_{em}}{\mathrm{d}t} = i_1 \cdot \frac{\mathrm{d}\phi_1}{\mathrm{d}t} + \cdots + i_n \cdot \frac{\mathrm{d}\phi_n}{\mathrm{d}t} - T_{em} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

The **Electromagnetic Energy** stored in the machine is a **State Function**

It depends on the **State Variables** of the system

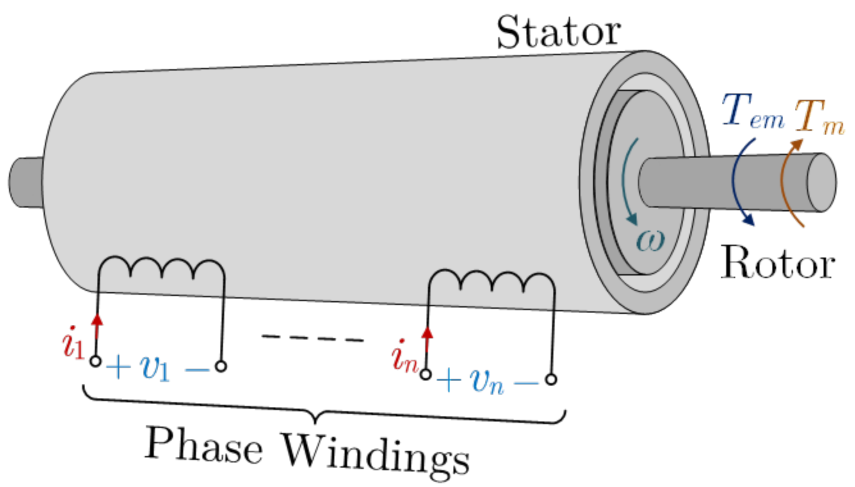
$$W_{em} = W_{em}\left(\phi_1, \ldots, \phi_n, \theta\right)$$

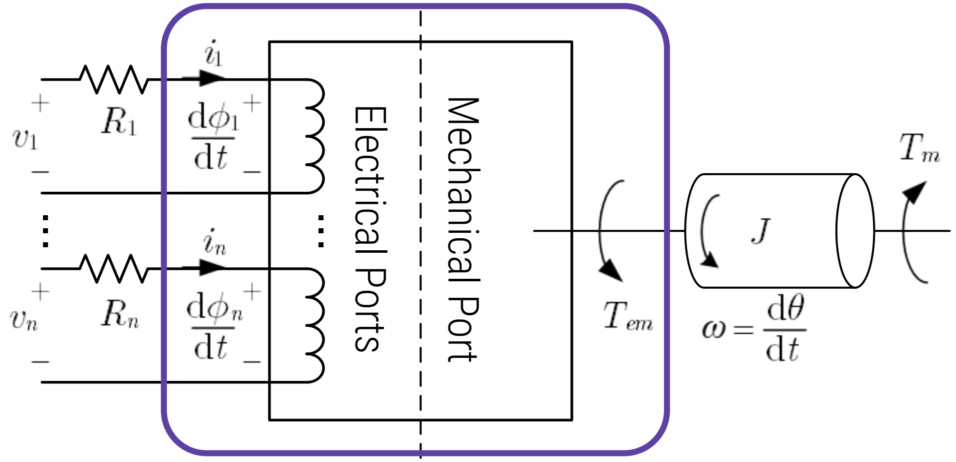
The time derivative can be computed with the chain rule

$$\frac{\mathrm{d}W_{em}}{\mathrm{d}t} = \frac{\partial W_{em}}{\partial \phi_1} \cdot \frac{\mathrm{d}\phi_1}{\mathrm{d}t} + \cdots + \frac{\partial W_{em}}{\partial \phi_n} \cdot \frac{\mathrm{d}\phi_n}{\mathrm{d}t} + \frac{\partial W_{em}}{\partial \theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

By comparing the two expressions, we obtain

$$i_1 = \frac{\partial W_{em}}{\partial \phi_1} \quad \cdots \quad i_n = \frac{\partial W_{em}}{\partial \phi_n} \qquad T_{em} = -\frac{\partial W_{em}}{\partial \theta}$$





ENERGY-BASED ELECTROMECHANICAL MODEL

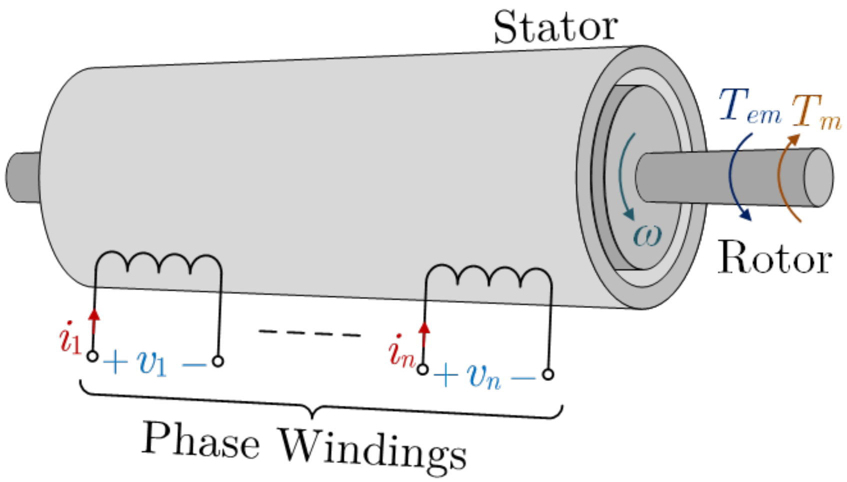
By energy conservation principle

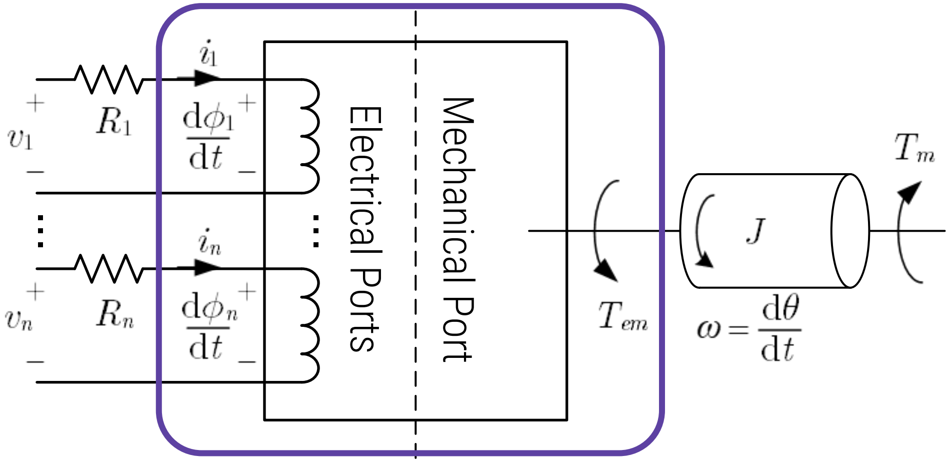
$$T_{em} = -\frac{\partial W_{em}}{\partial \theta}$$

$$i_1 = \frac{\partial W_{em}}{\partial \phi_1} \quad \cdots \quad i_n = \frac{\partial W_{em}}{\partial \phi_n}$$

If we find a closed-form expression of the electromagnetic energy

$$W_{em} = W_{em}\left(\phi_1, \ldots, \phi_n, \theta\right)$$



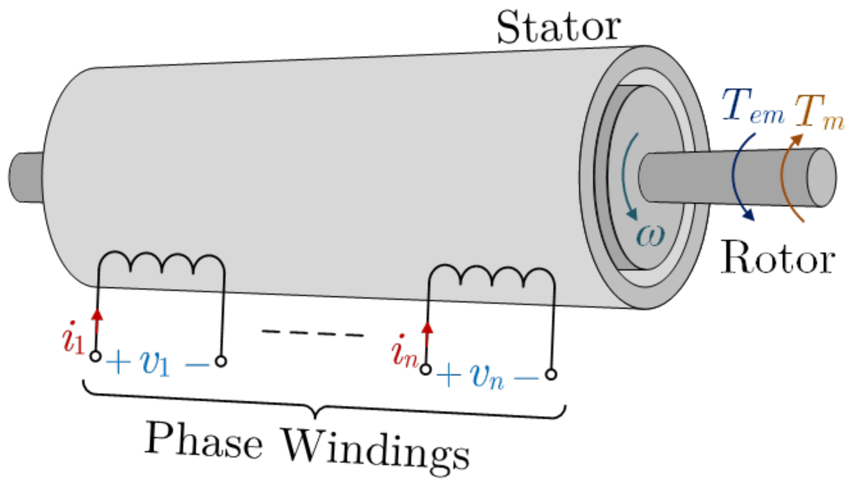


- The electromagnetic torque is the derivative of the energy with respect to the rotor angle (with negative sign)
- The phase currents are the derivative of the energy with respect to the flux linkages
- ? But we want the expressions of the fluxes

COENERGY-BASED ELECTROMECHANICAL MODEL

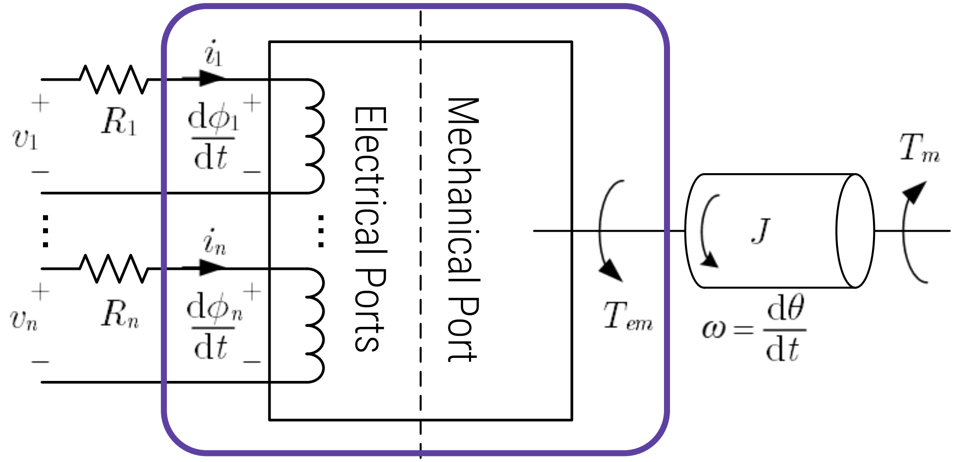
We define another **State Function**, named **Electromagnetic Coenergy**

$$W'_{em} = \boldsymbol{i}^{\mathrm{T}} \cdot \boldsymbol{\phi} - W_{em}$$
$$= i_1 \cdot \phi_1 + \cdots + i_n \cdot \phi_n - W_{em}$$



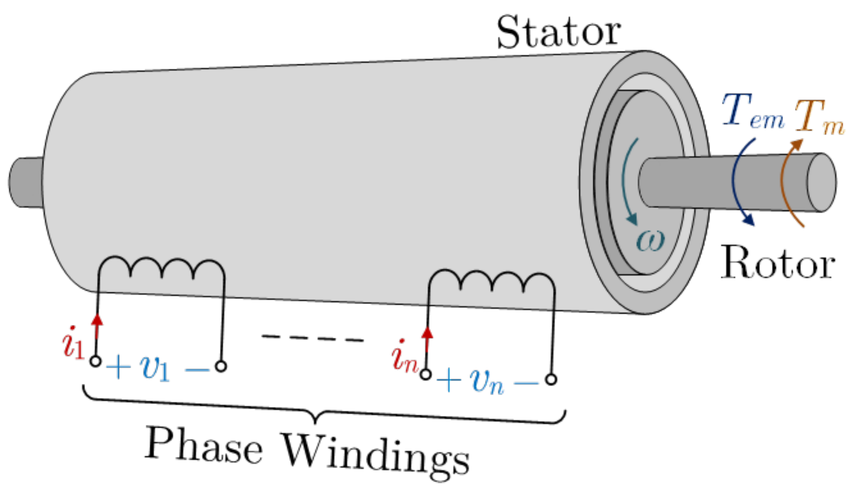
The time derivative of the coenergy can be found as:

$$\frac{\mathrm{d}W'_{em}}{\mathrm{d}t} = i_1 \cdot \frac{\mathrm{d}\phi_1}{\mathrm{d}t} + \cdots + i_n \cdot \frac{\mathrm{d}\phi_n}{\mathrm{d}t} + \cdots$$
$$\cdots + \frac{\mathrm{d}i_1}{\mathrm{d}t} \cdot \phi_1 + \cdots + \frac{\mathrm{d}i_n}{\mathrm{d}t} \cdot \phi_n - \frac{\mathrm{d}W_{em}}{\mathrm{d}t}$$
$$= \frac{\mathrm{d}i_1}{\mathrm{d}t} \cdot \phi_1 + \cdots + \frac{\mathrm{d}i_n}{\mathrm{d}t} \cdot \phi_n + T_{em} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$



We define another **State Function**, named **Electromagnetic Coenergy**

$$\frac{\mathrm{d}W'_{em}}{\mathrm{d}t} = \frac{\mathrm{d}i_1}{\mathrm{d}t} \cdot \phi_1 + \cdots + \frac{\mathrm{d}i_n}{\mathrm{d}t} \cdot \phi_n + T_{em} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$



The coenergy is, by definition, a function of currents and angle

$$W'_{em} = W'_{em}\left(i_1, \ldots, i_n, \theta\right)$$

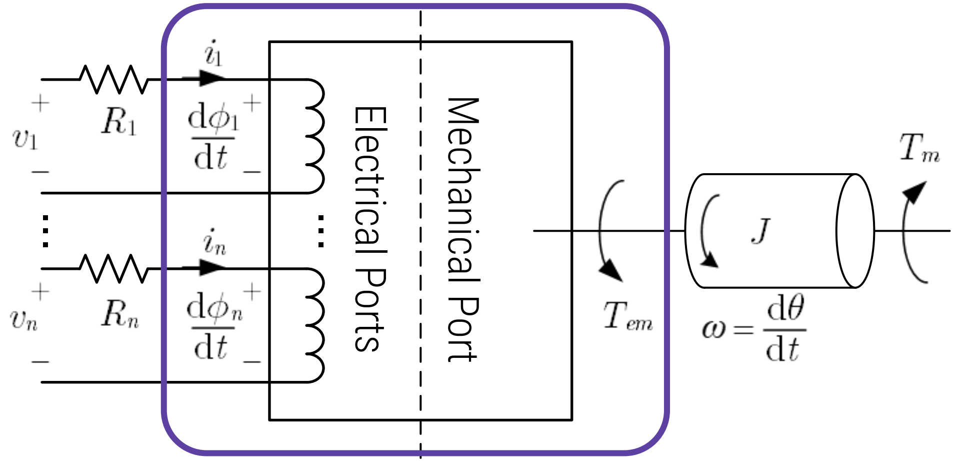
(we have changed the state variables of the system)

The time derivative can be computed with the chain rule

$$\frac{\mathrm{d}W'_{em}}{\mathrm{d}t} = \frac{\partial W'_{em}}{\partial i_1} \cdot \frac{\mathrm{d}i_1}{\mathrm{d}t} + \cdots + \frac{\partial W'_{em}}{\partial i_n} \cdot \frac{\mathrm{d}i_n}{\mathrm{d}t} + \frac{\partial W'_{em}}{\partial \theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

By comparing the two expressions, we obtain

$$\phi_1 = \frac{\partial W'_{em}}{\partial i_1} \quad \cdots \quad \phi_n = \frac{\partial W'_{em}}{\partial i_n} \qquad T_{em} = \frac{\partial W'_{em}}{\partial \theta}$$

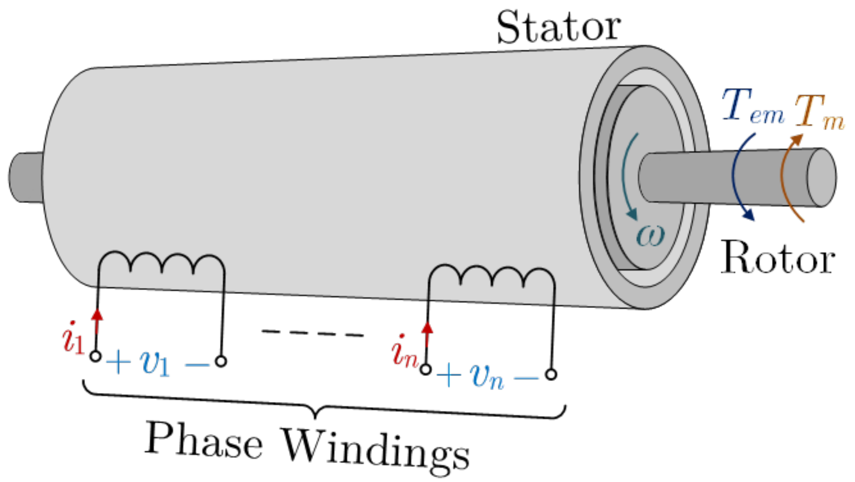


COENERGY-BASED ELECTROMECHANICAL MODEL

By introducing the **Electromagnetic Coenergy**, we obtain:

$$T_{em} = \frac{\partial W'_{em}}{\partial \theta}$$

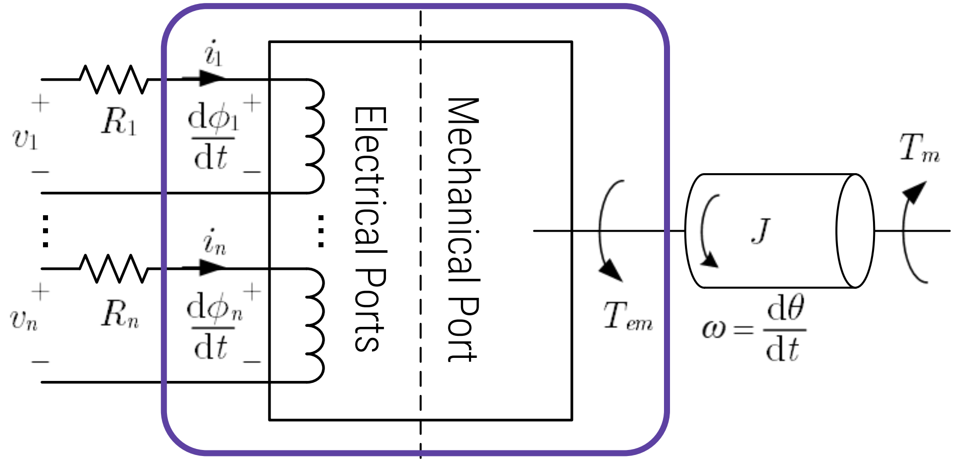
$$\phi_1 = \frac{\partial W'_{em}}{\partial i_1} \quad \cdots \quad \phi_n = \frac{\partial W'_{em}}{\partial i_n}$$



If we find a closed-form expression of the electromagnetic coenergy

$$W'_{em} = W'_{em}\left(i_1, \ldots, i_n, \theta\right)$$

- The electromagnetic torque is the derivative of the coenergy with respect to the rotor angle (no negative sign)
- The flux linkages are the derivative of the coenergy with respect to the phase currents



ENERGY AND COENERGY COMPARISONS

By using the Electromagnetic Energy:

$$W_{em} = W_{em}\underbrace{(\phi_1, \ldots, \phi_n}, \theta)$$

(Note: function of the flux linkages)

The electromagnetic torque is

$$T_{em} = -\frac{\partial W_{em}}{\partial \theta}$$

The currents are found from the flux linkages

$$i_1 = \frac{\partial W_{em}}{\partial \phi_1} \quad \cdots \quad i_n = \frac{\partial W_{em}}{\partial \phi_n}$$

By using the Electromagnetic Coenergy:

$$W'_{em} = W'_{em}\underbrace{(i_1, \ldots, i_n}, \theta)$$

(Note: function of the currents)

The electromagnetic torque is

$$T_{em} = \frac{\partial W'_{em}}{\partial \theta}$$

The flux linkages are found from the currents

$$\phi_1 = \frac{\partial W'_{em}}{\partial i_1} \quad \cdots \quad \phi_n = \frac{\partial W'_{em}}{\partial i_n}$$

➜ These results are of general validity

? We need to find the analytical expressions of energy and coenergy

MAGNETIC MODEL

A small insight in the electromagnetism theory

ENERGY AND COENERGY DENSITIES

? We need to find the analytical expressions of energy and coenergy

In the volume occupied by the machine:

$$W_{em} = \iiint_{V_{em}} \underbrace{w_{em}}_{} \, \mathrm{d}V$$

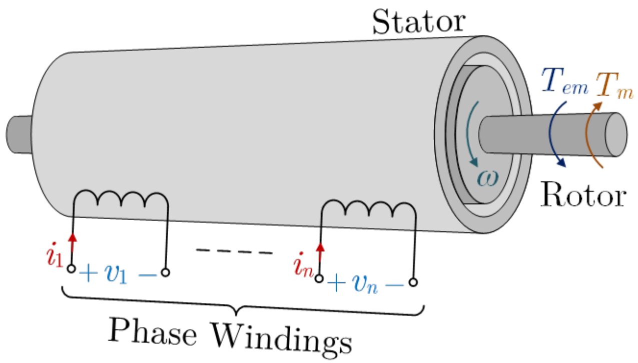
Electromagnetic Energy Density

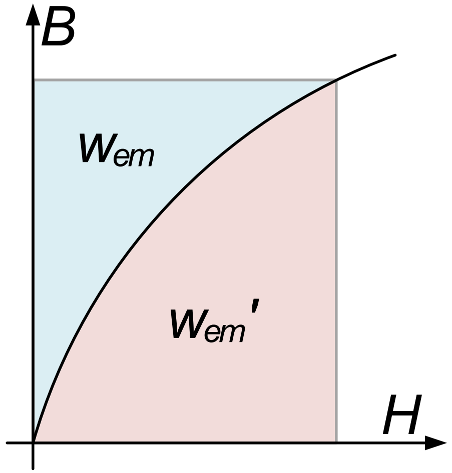
$$W'_{em} = \iiint_{V_{em}} \underbrace{w'_{em}}_{} \, \mathrm{d}V$$

Electromagnetic Coenergy Density

$$w_{em} = \int \vec{H}(\vec{B}) \cdot \mathrm{d}\vec{B}$$

$$w'_{em} = \int \vec{B}(\vec{H}) \cdot \mathrm{d}\vec{H}$$





➡ The energy and coenergy density depend (in each point of space) on the B-H characteristics of the material

MATERIALS IN ELECTRIC MACHINES

We need to examine different materials in the machine

$$w_{em} = \int \vec{H}(\vec{B}) \cdot \mathrm{d}\vec{B}$$

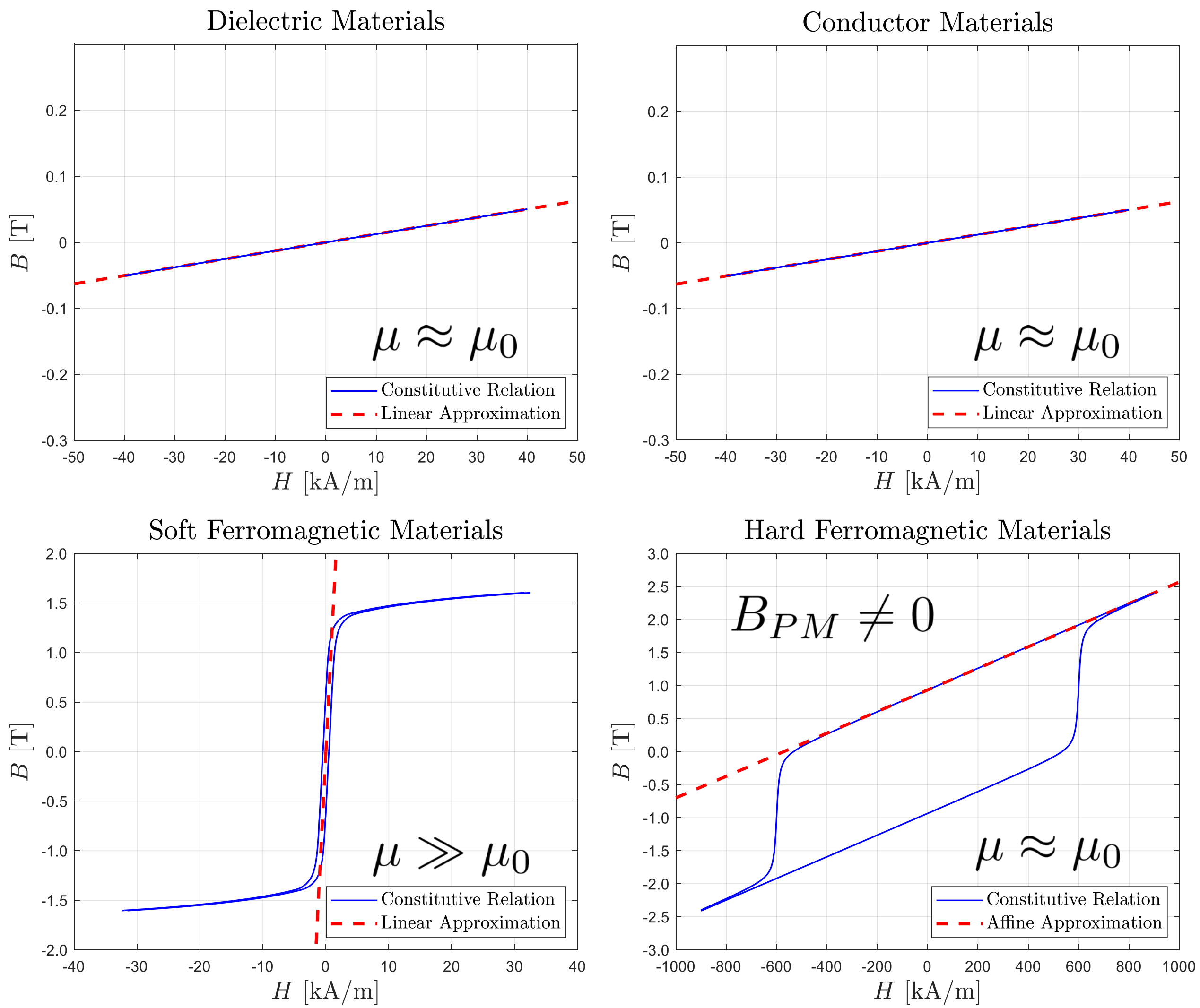
$$w'_{em} = \int \vec{B}(\vec{H}) \cdot \mathrm{d}\vec{H}$$

The materials in electric machines can be distinguished in:

- **Dielectric Materials** (air and insulating materials)
- **Conductor Materials** (copper or alluminum for windings)
- **Soft Ferromagnetic Materials** (iron for stator/rotor cores)
- **Hard Ferromagnetic Materials** (permanent magnets)

For simplicity, their **B-H characteristic is linearized**

$$\vec{B} = \underbrace{\mu}_{\text{Magnetic Permeability}} \vec{H} + \underbrace{\vec{B}_{PM}}_{\text{Permanent Magnetization}}$$ (affine characteristics)



ENERGY AND COENERGY IN LINEAR (OR AFFINE) MATERIALS

For simplicity, their **B-H characteristic is linearized**

$$\vec{B} = \mu\vec{H} + \vec{B}_{PM}$$ (affine characteristics)

The **electromagnetic energy density** is simplified to

$$w_{em} = \int \vec{H}\cdot \mathrm{d}\vec{B} = \int \vec{H}\cdot \mathrm{d}(\mu\vec{H} + \vec{B}_{PM}) = \int \mu\vec{H}\cdot \mathrm{d}\vec{H} \;\; = \frac{1}{2}\mu H^2$$

The **electromagnetic coenergy density** is simplified to

$$w'_{em} = \int \vec{B}\cdot \mathrm{d}\vec{H} = \int \vec{B}\cdot \mathrm{d}\left(\frac{\vec{B}}{\mu} - \frac{\vec{B}_{PM}}{\mu}\right) = \int \frac{1}{\mu}\vec{B}\cdot \mathrm{d}\vec{B} \;\; = \frac{1}{2}\frac{B^2}{\mu}$$

? We need to find an expression of B and H in each point of space

➜ Energy and Coenergy densities are equal in absence of permanent magnetization (in the linear hypothesis)

⚠ They are different in presence of permanent magnetization

MAGNETOQUASISTATIC MODEL OF THE MACHINE

$$w_{em} = \frac{1}{2}\mu H^2$$

$$w'_{em} = \frac{1}{2}\frac{B^2}{\mu}$$

? We need to find an expression of B and H in each point of space

For each rotor position θ, the fields B and H can be found by solving the magnetoquasistatic (MQS) problem:

$$\begin{cases} \oiint_{S_\tau} \vec{B}\cdot\hat{n}\,\mathrm{d}S = 0 & \text{(Gauss's law)} \\ \oint_\gamma \vec{H}\cdot\hat{t}\,\mathrm{d}l = \iint_{S_\gamma} \vec{J}_f\cdot\hat{n}\,\mathrm{d}S & \text{(Ampere's law)} \\ \vec{B} = \mu\vec{H} + \vec{B}_{PM} & \text{(material characteristics)} \\ \vec{J}_f = \sum_{k=1}^{n} i_k \cdot \vec{j}_{f,k} & \text{(distribution of windings currents)} \end{cases}$$

The system is linear → Superposition Principle

$$\vec{H} = \underbrace{\vec{H}^{(0)}}_{\text{Contribution of the Permanent Magnets}} + \sum_{k=1}^{n} \underbrace{\vec{h}^{(k)}\cdot i_k}_{\text{Contribution of the Phase Currents}}$$

$$\vec{B} = \vec{B}^{(0)} + \sum_{k=1}^{n} \vec{b}^{(k)}\cdot i_k$$

(Note: each coefficient depends on the rotor position θ)

TOTAL ENERGY AND COENERGY IN THE MACHINE

$$\vec{H} = \vec{H}^{(0)} + \sum_{k=1}^{n} \vec{h}^{(k)} \cdot i_k \quad \Rightarrow \quad w_{em} = \frac{1}{2}\mu H^2 \quad \Rightarrow \quad W_{em} = \iiint_{V_{em}} w_{em}\, \mathrm{d}V$$

$$\vec{B} = \vec{B}^{(0)} + \sum_{k=1}^{n} \vec{b}^{(k)} \cdot i_k \quad \Rightarrow \quad w'_{em} = \frac{1}{2}\frac{B^2}{\mu} \quad \Rightarrow \quad W'_{em} = \iiint_{V_{em}} w'_{em}\, \mathrm{d}V$$

The **total energy and coenergy** in the machine are:

$$W_{em} = \underbrace{W_{em}^{(0)}}_{\text{Contribution independent from the currents}} + \underbrace{\sum_{k=1}^{n} \left[W_{em,k}^{(1)} \cdot i_k\right]}_{\text{Contribution linearly varying with the currents}} + \underbrace{\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left[W_{em,k_1,k_2}^{(2)} \cdot i_{k_1} \cdot i_{k_2}\right]}_{\text{Contribution quadratically varying with the currents}}$$

$$W'_{em} = W'^{(0)}_{em} + \sum_{k=1}^{n} \left[W'^{(1)}_{em,k} \cdot i_k\right] + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left[W'^{(2)}_{em,k_1,k_2} \cdot i_{k_1} \cdot i_{k_2}\right]$$

(Note: each coefficient depends on the rotor position θ)

➡ We have found a closed form of the energy and coenergy in the electric machine

GENERAL MACHINE MODEL

Generalizing the results

ENERGY AND COENERGY COMPARISONS

By using the Electromagnetic Energy:

$$W_{em} = W_{em}\underbrace{(\phi_1, \dots, \phi_n}, \theta)$$

(Note: function of the flux linkages)

The electromagnetic torque is

$$T_{em} = -\frac{\partial W_{em}}{\partial \theta}$$

The currents are found from the flux linkages

$$i_1 = \frac{\partial W_{em}}{\partial \phi_1} \quad \cdots \quad i_n = \frac{\partial W_{em}}{\partial \phi_n}$$

By using the Electromagnetic Coenergy:

$$W'_{em} = W'_{em}\underbrace{(i_1, \dots, i_n}, \theta)$$

(Note: function of the currents)

The electromagnetic torque is

$$T_{em} = \frac{\partial W'_{em}}{\partial \theta}$$

The flux linkages are found from the currents

$$\phi_1 = \frac{\partial W'_{em}}{\partial i_1} \quad \cdots \quad \phi_n = \frac{\partial W'_{em}}{\partial i_n}$$

FLUX LINKAGES EXPRESSION

By using the Electromagnetic Coenergy:

$$W'_{em} = W'_{em}\left(i_1, \ldots, i_n, \theta\right) = W'^{(0)}_{em}(\theta) + \sum_{k=1}^{n}\left[W'^{(1)}_{em,k}(\theta) \cdot i_k\right] + \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\left[W'^{(2)}_{em,k_1,k_2}(\theta) \cdot i_{k_1} \cdot i_{k_2}\right]$$

The flux linkages are found from the currents

$$\phi_1 = \frac{\partial W'_{em}}{\partial i_1} = W'^{(1)}_{em,1}(\theta) + \sum_{k_2=1}^{n}\left[2W'^{(2)}_{em,1,k_2}(\theta) \cdot i_{k_2}\right] = \psi_{PM,1}(\theta) + \sum_{k_2=1}^{n}\left[L_{1,k_2}(\theta) \cdot i_{k_2}\right]$$

$$\ldots$$

$$\phi_n = \frac{\partial W'_{em}}{\partial i_n} = W'^{(1)}_{em,n}(\theta) + \sum_{k_2=1}^{n}\left[2W'^{(2)}_{em,n,k_2}(\theta) \cdot i_{k_2}\right] = \psi_{PM,n}(\theta) + \sum_{k_2=1}^{n}\left[L_{n,k_2}(\theta) \cdot i_{k_2}\right]$$

FLUX LINKAGES EXPRESSION

By using the Electromagnetic Coenergy:

$$W'_{em} = W'_{em}\left(i_1, \ldots, i_n, \theta\right) = W'^{(0)}_{em}(\theta) + \sum_{k=1}^{n}\left[\psi_{PM,k}(\theta) \cdot i_k\right] + \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\left[L_{k_1,k_2}(\theta) \cdot i_{k_1} \cdot i_{k_2}\right]$$

The flux linkages are found from the currents

$$\phi_1 = \frac{\partial W'_{em}}{\partial i_1} = \psi_{PM,1}(\theta) + \sum_{k_2=1}^{n}\left[L_{1,k_2}(\theta) \cdot i_{k_2}\right]$$

$$\cdots$$

$$\phi_n = \frac{\partial W'_{em}}{\partial i_n} = \underbrace{\psi_{PM,n}(\theta)}_{\text{Contribution independent from the currents}} + \sum_{k_2=1}^{n}\underbrace{\left[L_{n,k_2}(\theta) \cdot i_{k_2}\right]}_{\text{Contribution linearly varying with the currents}}$$

$$\overbrace{\boldsymbol{\phi}}^{\text{Total flux linkages}} = \underbrace{\boldsymbol{\psi}_{PM}(\theta)}_{\text{PM Induced Flux Linkages}} + \underbrace{\overbrace{\boldsymbol{L}}^{\text{Inductance Matrix}}(\theta) \cdot \boldsymbol{i}}_{\text{Currents Induced Flux Linkages}}$$

(Note: each coefficient depends on the rotor position θ)

(Note: the Inductance Matrix is **Symmetric** and **Positive Definite**)

INDUCED BACK–EMFS EXPRESSION

By using the Electromagnetic Coenergy:

$$W'_{em} = W'_{em}\left(i_1, \ldots, i_n, \theta\right) = W'^{(0)}_{em}(\theta) + \sum_{k=1}^{n} \left[\psi_{PM,k}(\theta) \cdot i_k\right] + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left[L_{k_1,k_2}(\theta) \cdot i_{k_1} \cdot i_{k_2}\right]$$

The flux linkages are expressed as:

$$\boldsymbol{\phi} = \boldsymbol{\psi}_{PM}(\theta) + \boldsymbol{L}(\theta) \cdot \boldsymbol{i}$$

From the electrical equations:

$$\boldsymbol{v} = \boldsymbol{R} \cdot \boldsymbol{i} + \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = \boldsymbol{R} \cdot \boldsymbol{i} + \frac{\mathrm{d}\boldsymbol{\psi}_{PM}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} \cdot \boldsymbol{i} + \boldsymbol{L}(\theta) \cdot \frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t}$$

$$= \underbrace{\boldsymbol{R} \cdot \boldsymbol{i}}_{\text{Resistive Drop}} + \underbrace{\omega \cdot \left[\frac{\partial \boldsymbol{\psi}_{PM}}{\partial \theta} + \frac{\partial \boldsymbol{L}}{\partial \theta} \cdot \boldsymbol{i}\right]}_{\text{Motional-Induced Back-EMFs}} + \underbrace{\boldsymbol{L}(\theta) \cdot \frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t}}_{\text{Transformer-Induced Back-EMFs}}$$

ELECTROMAGNETIC TORQUE EXPRESSION

By using the Electromagnetic Coenergy:

$$W'_{em} = W'_{em}\left(i_1, \ldots, i_n, \theta\right) = W'^{(0)}_{em}(\theta) + \sum_{k=1}^{n} \left[\psi_{PM,k}(\theta) \cdot i_k\right] + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left[L_{k_1,k_2}(\theta) \cdot i_{k_1} \cdot i_{k_2}\right]$$

$$= W'^{(0)}_{em}(\theta) \; + \; \boldsymbol{\psi}^{\mathrm{T}}_{PM}(\theta) \cdot \boldsymbol{i} \; + \; \frac{1}{2} \boldsymbol{i}^{\mathrm{T}} \cdot \boldsymbol{L}(\theta) \cdot \boldsymbol{i}$$

The electromagnetic torque is

$$T_{em} = \frac{\partial W'_{em}}{\partial \theta} = \frac{\partial W'^{(0)}_{em}}{\partial \theta} + \sum_{k=1}^{n} \left[\frac{\partial \psi_{PM,k}}{\partial \theta} \cdot i_k\right] + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left[\frac{\partial L_{k_1,k_2}}{\partial \theta} \cdot i_{k_1} \cdot i_{k_2}\right]$$

$$= \underbrace{\frac{\partial W'^{(0)}_{em}}{\partial \theta}}_{} + \underbrace{\frac{\partial \boldsymbol{\psi}^{\mathrm{T}}_{PM}}{\partial \theta} \cdot \boldsymbol{i}}_{} + \underbrace{\frac{1}{2} \boldsymbol{i}^{\mathrm{T}} \cdot \frac{\partial \boldsymbol{L}}{\partial \theta} \cdot \boldsymbol{i}}_{}$$

Contribution independent from the currents	Contribution linearly varying with the currents	Contribution quadratically varying with the currents
Magnets only related torque	Interaction Currents/Magnets	Interaction Currents/Currents

ELECTROMAGNETIC TORQUE PRODUCTION MECHANISMS

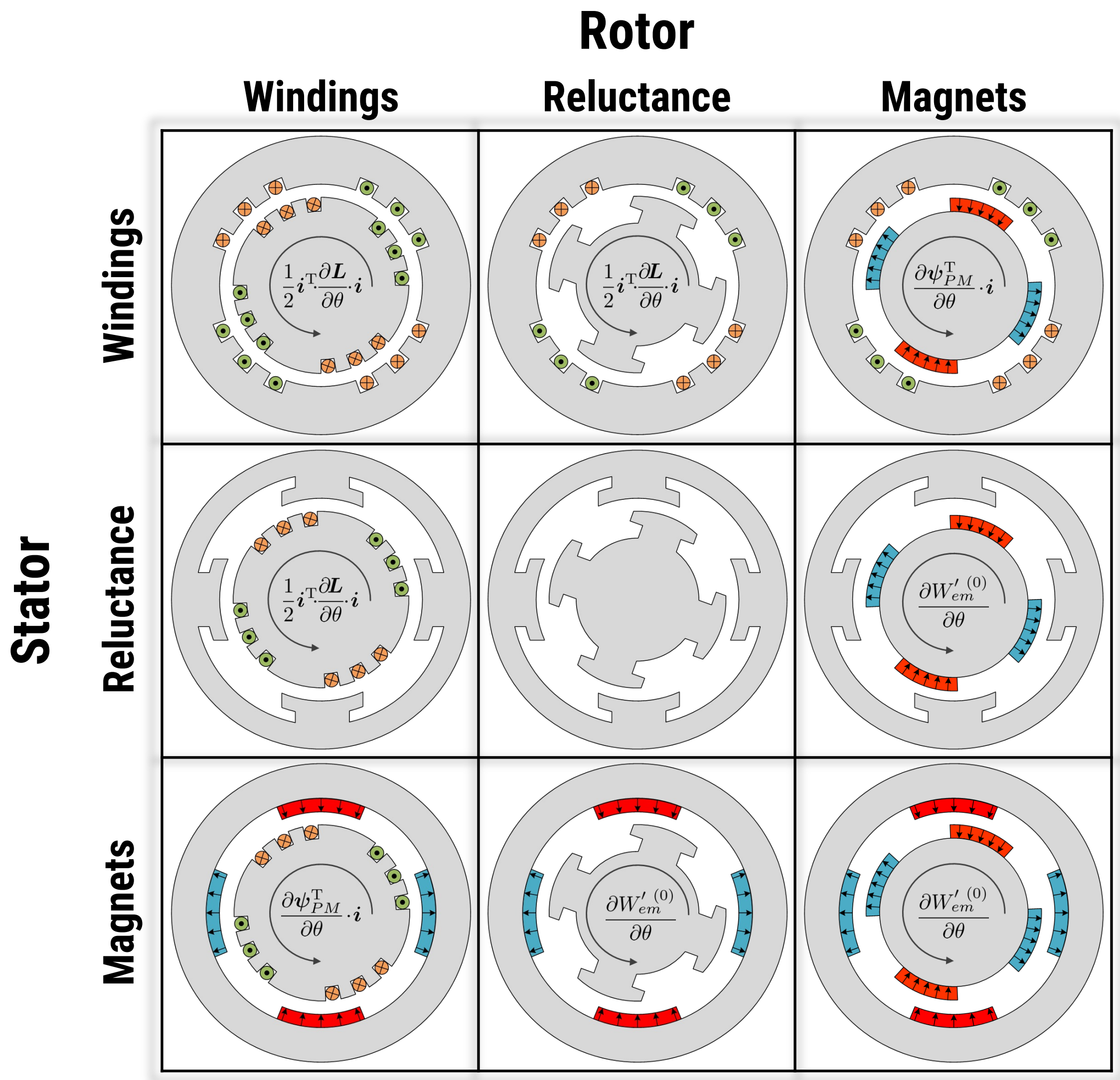
The electromagnetic torque is

$$T_{em} = \underbrace{\frac{\partial W'^{(0)}_{em}}{\partial \theta}}_{\text{Magnets only related torque}} + \underbrace{\frac{\partial \boldsymbol{\psi}^{\mathrm{T}}_{PM}}{\partial \theta} \cdot \boldsymbol{i}}_{\text{Interaction Currents/Magnets}} + \underbrace{\frac{1}{2} \boldsymbol{i}^{\mathrm{T}} \cdot \frac{\partial \boldsymbol{L}}{\partial \theta} \cdot \boldsymbol{i}}_{\text{Interaction Currents/Currents}}$$

We can classify different torque development mechanisms according to the **Stator and Rotor interactions**

On either the stator or the rotor we can use:

- **Windings**
- **Variable Reluctance**
- **Permanent Magnets**

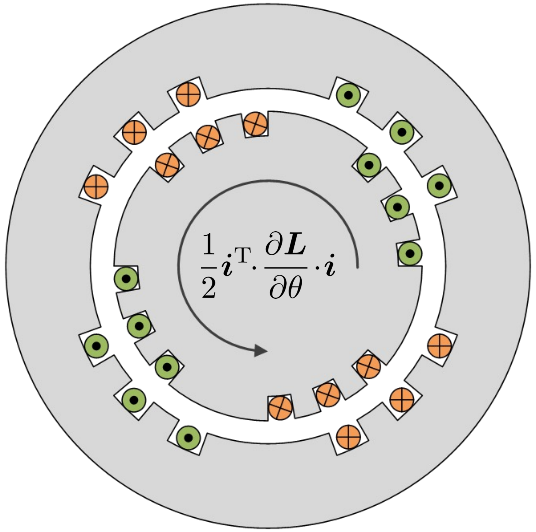


Multiple mechanisms can coexist

CONDITIONS FOR GENERATING NON-ZERO TORQUE

Generally, for a two-winding machine, only the stator-rotor inductances are variable

$$T_{em} = \frac{1}{2}\boldsymbol{i}^{\mathrm{T}} \cdot \frac{\partial \boldsymbol{L}}{\partial \theta} \cdot \boldsymbol{i} = \frac{1}{2}\begin{bmatrix}\boldsymbol{i}_s & \boldsymbol{i}_r\end{bmatrix} \cdot \frac{\partial}{\partial \theta}\begin{bmatrix}\boldsymbol{L}_s & \boldsymbol{L}_m(\theta) \\ \boldsymbol{L}_m^{\mathrm{T}}(\theta) & \boldsymbol{L}_r\end{bmatrix} \cdot \begin{bmatrix}\boldsymbol{i}_s \\ \boldsymbol{i}_r\end{bmatrix} = \underbrace{\boldsymbol{i}_s^{\mathrm{T}}}_{\omega_s} \cdot \underbrace{\frac{\partial \boldsymbol{L}_m}{\partial \theta}}_{p\omega_m} \cdot \underbrace{\boldsymbol{i}_r}_{\omega_r}$$

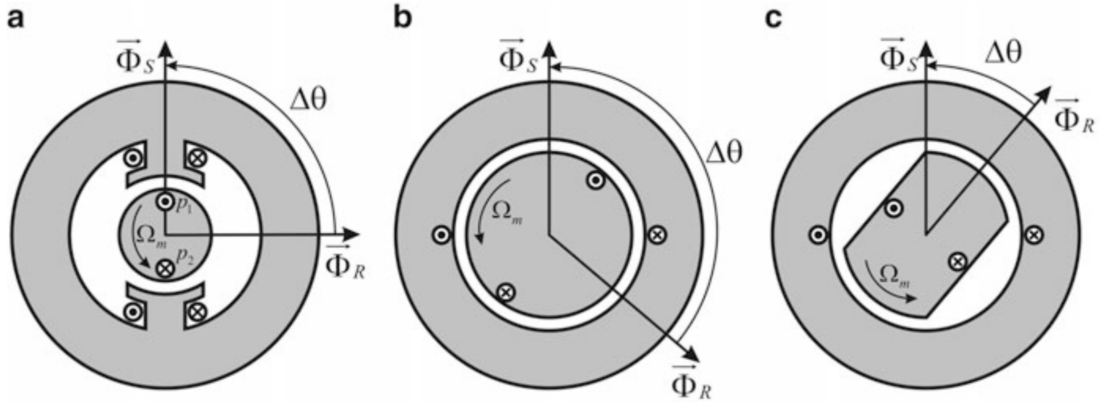


Then, we can distinguish:

- stator current frequency - ω_s
- rotor current frequency - ω_r
- rotor mechanical speed - ω_m
- rotor electrical speed - $p\omega_m$
  (With p being the Pole Pairs of the machine)

The condition for non-zero average torque generation is:

$$\omega_s \pm \omega_r \pm p\,\omega_m = 0$$



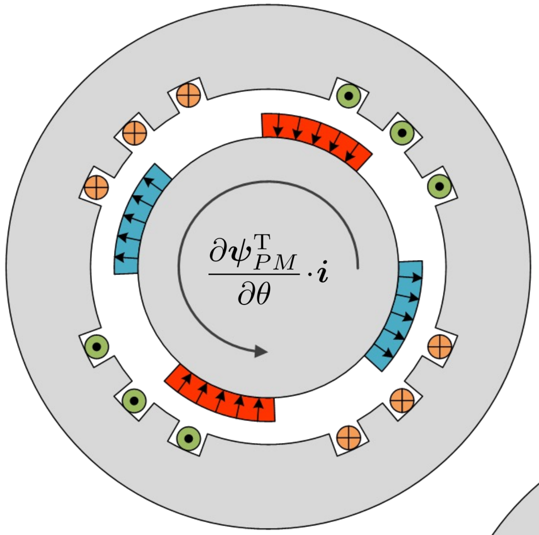
Different kinds of machines have different speeds:

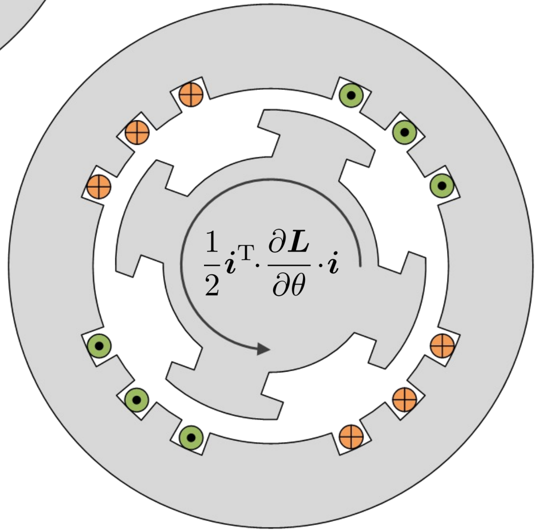
- **DC Machines**: $\omega_s = 0$, $\omega_r = p \cdot \omega_m$
- **AC Synchronous Machines**: $\omega_r = 0$, $\omega_s = p \cdot \omega_m$
- **AC Asynchronous Machines**: $\omega_s - p \cdot \omega_m = \omega_r$

CONDITIONS FOR GENERATING NON-ZERO TORQUE

Generally, for a single-winding machine based on permanent magnets or on variable reluctance

$$T_{em} = \underbrace{\frac{\partial \boldsymbol{\psi}_{PM}^{\mathrm{T}}}{\partial \theta}}_{p\omega_m} \cdot \boldsymbol{i} \qquad \text{or} \qquad T_{em} = \frac{1}{2} \underbrace{\boldsymbol{i}^{\mathrm{T}}}_{\omega_s} \cdot \underbrace{\frac{\partial \boldsymbol{L}}{\partial \theta}}_{2p\omega_m} \cdot \underbrace{\boldsymbol{i}}_{\omega_s}$$





Then, we can distinguish:

- stator current frequency - ω_s
- rotor mechanical speed - ω_m
- rotor electrical speed - $p\omega_m$
  (With p being the Pole Pairs of the machine)

The condition for non-zero average torque generation is:

$$\omega_s \pm p\,\omega_m = 0$$

Only DC and Synchronous machines are possible:

- **DC Machines**: $\omega_r = p \cdot \omega_m$ (windings are on the rotor)
- **AC Synchronous Machines**: $\omega_s = p \cdot \omega_m$

(Interior-Mounted Permanent Magnet Synchronous Machines use both torque generation mechanisms at the same time)

SUMMARY

General model of an electrical machine

GENERAL DYNAMICAL MODEL OF AN ELECTRICAL MACHINE

Electrical Equations

$$\begin{cases} \boldsymbol{v} = \boldsymbol{R} \cdot \boldsymbol{i} + \dfrac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = \boldsymbol{R} \cdot \boldsymbol{i} + \omega \cdot \left[\dfrac{\partial \boldsymbol{\psi}_{PM}}{\partial \theta} + \dfrac{\partial \boldsymbol{L}}{\partial \theta} \cdot \boldsymbol{i} \right] + \boldsymbol{L}(\theta) \cdot \dfrac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t} \\ \boldsymbol{\phi} = \boldsymbol{\psi}_{PM}(\theta) + \boldsymbol{L}(\theta) \cdot \boldsymbol{i} \end{cases}$$

Mechanical Equations

$$\begin{cases} \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \\ J \cdot \dfrac{\mathrm{d}\omega}{\mathrm{d}t} + F(\omega) \cdot \omega = T_{em} - T_m \\ T_{em} = \dfrac{\partial W'^{(0)}_{em}}{\partial \theta} + \dfrac{\partial \boldsymbol{\psi}^{\mathrm{T}}_{PM}}{\partial \theta} \cdot \boldsymbol{i} + \dfrac{1}{2} \boldsymbol{i}^{\mathrm{T}} \cdot \dfrac{\partial \boldsymbol{L}}{\partial \theta} \cdot \boldsymbol{i} \end{cases}$$